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Canonical Symmetry of a Constrained Hamiltonian System and Canonical Ward Identity

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An algorithm for the construction of the generators of the gauge transformation of a constrained Hamiltonian system is given. The relationships among the coefficients connecting the first constraints in the generator are made clear. Starting from the phase space generating function of the Green function, the Ward identity in canonical formalism is deduced. We point out that the quantum equations of motion in canonical form for a system with singular Lagrangian differ from the classical ones whether Dirac's conjecture holds true or not. Applications of the present formulation to the Abelian and non-Abelian gauge theories are given. The expressions for PCAC and generalized PCAC of the AVV vertex are derived exactly from another point of view. A new form of the Ward identity for gauge-ghost proper vertices is obtained which differs from the usual Ward–Takahashi identity arising from the BRS invariance.

1. INTRODUCTION

The discussion of symmetry of a system is usually based on examination of the Lagrangian in configuration space (coordinate space) and the corresponding transformation expressed in terms of Lagrange's variables. The system with a singular Lagrangian is subject to some inherent phase space constraint and is called a constrained Hamiltonian system. The classical canonical symmetry properties for a constrained Hamiltonian system are discussed in previous work (Li, 1991, 1993a). Here the quantum symmetry in the canonical formalism for a constrained Hamiltonian system is further investigated.

Dirac's theory of constrained systems plays an important role in modern quantum field theories. By using it, many of the central problems which

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appeared in the development of the quantization procedures of the gauge and gravitational fields have been solved. In spite of these general achievements some basic problems in this theory are still widely discussed in the literature. One of them is Dirac's (1964) conjecture. Dirac conjectured that all the first-class constraints are generators of the gauge transformation. From time to time there have been objections to Dirac's conjecture (Li, 1993b,c). But we do not know of any physically important system in which Dirac's conjecture leads to the wrong results.

The paper is organized as follows. In Section 2 an algorithm for the construction of the generator of the gauge transformation is discussed which is slightly different from the Castellani (1982) treatment, and a generalization to the field theory is given. A clear discussion is made to show that the time evolution of the coefficients of the secondary first-class constraints in the generator of the gauge transformation is determined by the coefficients of the primary first-class constraints (Galvão and Boechat, 1990). A comparison of both results (Castellani, 1982; Galvão and Boechat, 1990) is discussed. In Section 3, based on the invariance of the translation in canonical variables, the quantum equation of motion in canonical form for a constrained Hamiltonian system is derived. We point out that this quantum canonical equation of motion for a system with singular Lagrangian is different from the classical canonical equation of motion whether Dirac's conjecture holds true or not. In Section 4 the Ward identity on the canonical formalism for a constrained Hamiltonian system is deduced. What is different from the traditional formulation is that we do not carry out the integration over momenta. In Section 5, a preliminary application of our results to a model in field theory which is functionally equivalent to the mixed Chern–Simons theory is given, and the generator of the gauge transformation and some relations of proper vertices are found. In Section 6, an application of the present formulation to the non-Abelian gauge theory is illustrated, and the expressions for PCAC and generalized PCAC of the AVV vertex are derived. A comment is made to argue that the constraints had been ignored in deriving the Ward–Takahashi identity by some authors (Suura and Young, 1973). A new form of the Ward identity for gauge-ghost proper vertices is derived which differs from the usual Ward–Takahashi identity arising from the BRS invariance.

2. THE GENERATOR OF GAUGE TRANSFORMATION

Gauge theories play an important role in modern field theories. These theories have a gauge invariance under the local transformation (or gauge transformation). The algorithm for the construction of the gauge generator of the gauge transformation of a constrained Hamiltonian system has been discussed (Castellani, 1982; Galvão and Boechat, 1990; Henneaux *et al.*, 1990). This problem will be studied here starting from another point of view (Castellani, 1982).

Consider a system described by the field variables $\psi^\alpha(x)$ ($\alpha = 1, 2, \ldots, N$), $x = (t, \mathbf{x})$. The Lagrangian of the system depends on a set of the variables $\psi^\alpha(x)$ and their first-order derivatives: $\mathcal{L}(\psi^\alpha, \psi^\alpha_{,\mu})$, $\psi^\alpha_{,\mu} = \partial_\mu \psi^\alpha = \partial\psi^\alpha/\partial x^\mu$ ($\mu = 0, 1, 2, 3$). The flat space-time metric is $\eta_{\mu\nu} = \text{diag}(+ - - -)$. In many interesting physical system the Lagrangian is singular (for example, gauge theories), i.e., the Hessian matrix $H_{\alpha\beta}$ is degenerate

$$\det(H_{\alpha\beta}) = \det\left(\frac{\partial^2 \mathcal{L}}{\partial\dot{\psi}^\alpha \partial\dot{\psi}^\beta}\right) = 0 \tag{1}$$

The Legendre transformation introduces canonical momenta $\pi_\alpha = \partial\mathcal{L}/\partial\dot{\psi}^\alpha$; one can then go over from the Lagrangian description to the Hamiltonian description. It is supposed that the rank of the Hessian matrix is $N - R$; then one cannot solve all $\dot{\psi}^\alpha$ from the definition of canonical momenta because of (1). This implies the existence of constraints

$$\phi^0_a(\psi^\alpha, \pi_\alpha) \approx 0 \qquad (a = 1, 2, \ldots, R) \tag{2}$$

where the sign $\approx$ (weak equality) means equality on the constrained hypersurface. Equation (2) is called the primary constraint. The classical equations of motion of the Hamiltonian system are given by (Gitman and Tyutin, 1990)

$$\dot{\psi}^\alpha \approx \frac{\delta H_T}{\delta \pi_\alpha} \approx \{\psi^\alpha, H_T\}, \qquad \dot{\pi}_\alpha \approx -\frac{\delta H_T}{\delta\psi^\alpha} \approx \{\pi_\alpha, H_T\} \tag{3}$$

where H_T is the total Hamiltonian, $H_T = \int d^3\, x(\mathcal{H}_c + \lambda^a \phi^0_a)$, $\mathcal{H}_c$ is the canonical Hamiltonian density, $\mathcal{H}_c = \pi_\alpha\dot{\psi}^\alpha - \mathcal{L}$, $\lambda^a = \lambda^a(x)$ are Lagrange multipliers, and $\{\cdot, \cdot\}$ denotes the Poisson bracket in field theories. Using the stationarity conditions of primary constraints, $\{\phi^0_a, H_T\} \approx 0$, one can define successively the secondary constraints according to the Dirac–Bergmann algorithm

$$\phi^k_a \approx \{\phi^{k-1}_a, H_T\} \tag{4}$$

This algorithm is continued until ϕ^m_a satisfies

$$\phi^{m+1}_a \approx \{\phi^m_a, H_T\} = C^b_{ak}\phi^k_b \qquad (k \le m) \tag{5}$$

All the constraints ϕ are classified into two classes. A ϕ_a is defined to be of first class if $\{\phi_a, \phi_b\} = 0$ (mod ϕ_c) for all ϕ_b. Otherwise it is of second class.

For the sake of simplicity, all the constraints of the system are assumed to be of first class. Under an infinitesimal gauge transformation suppose the two trajectories $(\psi^\alpha(x), \pi_\alpha(x))$ and $(\psi^\alpha(x) + \delta\psi^\alpha(x), \pi_\alpha(x) + \delta\pi_\alpha(x))$ both satisfy the constraint conditions (2) and equations of motion (3); then varied trajectory equations (3) and constraint conditions (2) can be expanded to first order in small variations of canonical variables, and using equations (2) and (3) for the unvaried trajectory, one finds

$$\delta\dot{\psi}^{\alpha} \approx \int d^3x \left[\frac{\delta^2 H_T}{\delta\psi^{\beta}\delta\pi_{\alpha}} \, \delta\psi^{\beta} + \frac{\delta^2 H_T}{\delta\pi_{\beta}\delta\pi_{\alpha}} \, \delta\pi_{\beta} \right] \tag{6a}$$

$$\delta\dot{\pi}_{\alpha} \approx -\int d^3x \left[\frac{\delta^2 H_T}{\delta\psi^{\beta}\delta\psi^{\alpha}} \, \delta\psi^{\beta} + \frac{\delta^2 H_T}{\delta\pi_{\beta}\delta\psi^{\alpha}} \, \delta\pi_{\beta} \right] \tag{6b}$$

$$\frac{\partial\phi_a^0}{\partial\psi^{\alpha}} \, \delta\psi^{\alpha} + \frac{\partial\phi_a^0}{\partial\pi_{\alpha}} \, \delta\pi_{\alpha} \approx 0 \tag{6c}$$

Now let the variations of canonical variables be generated by a phase space functional G and be parametrized by the arbitrary infinitesimal functions $\epsilon_j(x)$; we consider a generator of gauge transformation of the type

$$G = \int d^3x \, \epsilon_j^{(k)} G_k^j(\psi^{\alpha}, \pi_{\beta}) \tag{7}$$

where $\epsilon_j^{(k)} = \partial_0^k \epsilon_j(x)$, and $\epsilon_j(x)$ are independent arbitrary functions of time-space. The variations of $\psi^{\alpha}(x)$ and $\pi_{\alpha}(x)$ induced by G are given by

$$\delta\psi^{\alpha} = \{\psi^{\alpha}, G\} = \frac{\delta G}{\delta\pi_{\alpha}}, \qquad \delta\pi_{\alpha} = \{\pi_{\alpha}, G\} = -\frac{\delta G}{\delta\psi^{\alpha}} \tag{8}$$

Substituting (8) into (6), because of the arbitrariness of $\epsilon_j(x)$, one finds the following conditions on the G_k^j:

$$\frac{\partial}{\partial\psi^{\alpha}} \left[G_{k-1}^j + \{G_k^j, H_T\}\right] = 0 \quad (\text{mod } \phi_a^0) \tag{9a}$$

$$\frac{\partial}{\partial\pi_{\alpha}} \left[G_{k-1}^j + \{G_k^j, H_T\}\right] = 0 \quad (\text{mod } \phi_a^0) \tag{9b}$$

$$\{G_k^j, \phi_a^0\} = 0 \quad (\text{mod } \phi_a^0) \tag{9c}$$

Since we are considering the variations that leave the trajectory on the constraint hypersurface, we should add the further requirement that $\{G_k^j, \phi_a^n\} = 0$ (mod ϕ_a^0) for the secondary constraints ϕ_a^n to the third set of equations (9c). Thus, all the G_k^j have to be first-class constraints. The H_c can be substituted instead of the H_T, owing to the assumption that all the constraints are of first class. From (9) one finds the following recursive relations in a manner analogous to the discussion which was given by Castellani (1982):

$$\{G_0^j, H_0\} = 0 \quad (\text{mod } \phi_a^0) \tag{10a}$$

$$G_{k-1}^j + \{G_k^j, H_c\} = 0 \quad (\text{mod } \phi_a^0) \tag{10b}$$

$$G_m^j = 0 \quad (\text{mod } \phi_a^0) \tag{10c}$$

Therefore, all the G_k^j have to be first-class constraints and, with the exception of those first-class constraints which arise as powers χ^n (Castellani, 1982), are part of the gauge generator. The G_{k-1}^j is deduced from G_k^j according to the recursive relations (10b). Moreover, G_m^j must be a primary first-class constraint, for every primary first-class constraint using (10) to construct the chains of G_k^j until G_0^j is reached. After the G_k^j are found, the generator G of the gauge transformation can be constructed by using (7).

Even when second-class constraints appear, if the series of first-class constraints derived from primary first-class constraints are completely separated from the series of second-class ones, the above formulation to construct the generator of the gauge transformation is valid for such a system.

Galvão and Boechat (1990) discussed the relationships of the coefficients connected with the first-class constraints in the generator of the gauge transformation. But they discarded the last term on the left-hand side of equation (3.8) in their paper. Here is a simple treatment to obtain the fundamental results clearly. For the sake of simplicity, let us consider a system with finite degrees of freedom and suppose that the set of all independent constraints is of first class and split these constraints into primary ϕ_a^0 and secondary χ_a^k ones. According to Dirac's prescription, the generator of the gauge transformation for a constrained Hamiltonian system endowed with primary and secondary first-class constraints is constructed as a linear combination of all these first-class constraints. Thus the generator of the gauge transformation can be expressed as

$$G = \theta_a(t)\phi_a^0 + \omega_a^k(t)\chi_a^k \tag{11}$$

The generator G should satisfy the following conditions:

$$\frac{\partial G}{\partial t} + \{G, H_T\} = 0 \quad (\text{mod } \phi_a^0) \tag{12a}$$

$$\{G, \phi_a^0\} = 0 \quad (\text{mod } \phi_a^0) \tag{12b}$$

Equations (12a) and (12b) are necessary and sufficient conditions for G to be the generator of the gauge transformation (Sugano, 1990). Substituting (11) into (12), one gets

$$\theta_a\{\phi_a^0, H_c\} + \frac{d\omega_a^k}{dt}\chi_a^k + \omega_a^k\{\chi_a^k, H_c\} = 0 \quad (\text{mod } \phi_a^0) \tag{13}$$

The quantities $\{\phi_a^0, H_c\}$ and $\{\chi_a^k, H_c\}$ can be expressed as

$$\{\chi_a^k, H_c\} = \alpha_{ae'}^{kk'}\chi_{e'}^{k'} \pmod{\phi_a^0} \tag{14a}$$

$$\{\phi_a^0, H_c\} = \beta_{af}^k\chi_f^k \pmod{\phi_a^0} \tag{14b}$$

Substituting (14) into (13) and taking into account the linear independence of all constraints and from equation (12b), one obtains the following differential equations related to the coefficients $\theta_a(t)$ and $\omega_a^k(t)$ in the generator (11):

$$\frac{d\omega_a^k}{dt} + \alpha_{ae'}^{kk'}\omega_{e'}^{k'} + \beta_{af}^k\theta_f = 0 \pmod{\phi_a^0} \tag{15}$$

This result was obtained by Galvão and Boechat (1990). Equation (15) implies that the time evolution of the coefficients of the secondary constraints in the generator of the gauge transformation is not independent, but is determined by the coefficients of the primary constraints.

We apply these results to the case discussed by Castellani (1982). According to the Dirac–Bergmann algorithm of the constraints, from the primary constraints one can define successively the secondary constraints. Using equations (14) and (15), one finds

$$\dot{\omega}_a^1(t) + \theta_a(t) = 0 \tag{16a}$$

$$\dot{\omega}_a^k(t) + \theta_a^{k-1}(t) = 0 \tag{16b}$$

Let $\theta(t) = d^M\epsilon(t)/dt^M = \epsilon^{(M)}(t)$ $(M = m + 1)$; from (11) and (16) one obtains the generator of the gauge transformation as

$$G = \epsilon_a^{(M)}\phi_a^0 + (-1)^k\epsilon_a^{(k)}\chi_a^k \tag{17}$$

This result agrees with the conclusion of Castellani (1982).

3. TRANSLATION INVARIANCE OF GENERATING FUNCTIONAL IN EXTENDED PHASE SPACE

Let us first consider a system with regular (nonsingular) Lagrangian. Introducing the exterior sources $J_\alpha(x)$ and $J_1^\alpha(x)$ for the fields $\psi^\alpha(x)$ and the momenta $\pi_\alpha(x)$, respectively, we can write the generating functional of the Green function for this system as

$$Z[J, J_1] = \int \mathfrak{D}\psi^\alpha\, \mathfrak{D}\pi_\alpha \exp\left\{i\left[I^P + \int d^4x\, (J_\alpha\psi^\alpha + J_1^\alpha\pi_\alpha)\right]\right\} \tag{18a}$$

where I^P is a canonical action,

$$I^P = \int \mathcal{L}^P \, d^4x = \int (\pi_\alpha \dot{\psi}^\alpha - \mathcal{H}_c) \, d^4x \tag{18b}$$

The Green function $G(x_1, x_2, \ldots, x_n)$ is given by

$$\begin{aligned} G(x_1, x_2, \ldots, x_n) &= \langle 0 | T[\psi^\alpha(x_1)\psi^\alpha(x_2) \cdots \psi^\alpha(x_n)] | 0 \rangle \\ &= \frac{1}{i^n} \frac{\delta Z[J, J_1]}{\delta J_\alpha(x_1)\delta J_\alpha(x_2) \cdots \delta J_\alpha(x_n)} \bigg|_{J=J_1=0} \end{aligned} \tag{19}$$

Let us consider a special translation of $\psi^\alpha(x)$:

$$\begin{cases} \psi^{\alpha\prime}(x) = \psi^\alpha(x) + \epsilon^\alpha(x) \\ \quad \pi'_\alpha = \pi_\alpha(x) \end{cases} \tag{20}$$

where $\epsilon^\alpha(x)$ ($\alpha = 1, 2, \ldots, N$) are infinitesimal arbitrary functions and their value on the boundary of the time-space domain vanishes. The Jacobian of the transformation (20) is equal to unity. The generating functional (18a) is invariant under the transformation (20), thus

$$\begin{aligned} Z[J, J_1] = \int \mathcal{D}\psi^\alpha \, \mathcal{D}\pi_\alpha \left[1 + i \int d^4x \left(\frac{\delta I^P}{\delta \psi^\alpha} + J_\alpha \right) \epsilon^\alpha(x) \right] \\ \times \exp\left\{ i \left[I^P + \int d^4x \, (J_\alpha \psi^\alpha + J_1^\alpha \pi_\alpha) \right] \right\} \end{aligned} \tag{21}$$

where

$$\frac{\delta I^P}{\delta \psi^\alpha} = -\dot{\pi}_\alpha - \frac{\delta H_c}{\delta \psi^\alpha} \tag{22}$$

The translation invariance of the generating functional implies $\delta Z[J, J_1]/\delta\epsilon^\alpha |_{\epsilon^\alpha=0} = 0$; this leads to

$$\int \mathcal{D}\psi^\alpha \, \mathcal{D}\pi_\alpha \left(\frac{\delta I^P}{\delta \psi^\alpha} + J_\alpha \right) \exp\left\{ i \int d^4x \, (\mathcal{L}^P + J_\alpha \psi^\alpha + J_1^\alpha \pi_\alpha) \right\} = 0 \tag{23}$$

Differentiating (23) with respect to $J_\alpha(x)$ n times, one obtains

$$\begin{aligned} \int \mathcal{D}\psi^\alpha \, \mathcal{D}\pi_\alpha \Bigg[\left(\frac{\delta I^P}{\delta \psi^\alpha} + J_\alpha \right) \psi^{\alpha_1}(x_1) \cdots \psi^{\alpha_n}(x_n) \\ - i \sum_{\alpha_\beta} \delta_{\alpha\alpha_\beta} \delta(x - x_\beta) \psi^{\alpha_1}(x_1) \cdots \psi^{\alpha_{\beta-1}}(x_{\beta-1}) \psi^{\alpha_{\beta+1}}(x_{\beta+1}) \cdots \psi^{\alpha_n}(x_n) \Bigg] \\ \times \exp\left\{ i \int d^4x \, [\mathcal{L}^P + J_\alpha \psi^\alpha + J_1^\alpha \pi_\alpha] \right\} = 0 \end{aligned} \tag{24}$$

Let $J_\alpha = J_1^\alpha = 0$; one gets

$$\left\langle 0 \middle| \left[T^* \left[\left(\frac{\delta I^P}{\delta \psi^\alpha} \right) \psi^{\alpha_1}(x_1) \cdots \psi^{\alpha_n}(x_n) \right] \middle| 0 \right\rangle$$

$$= i \sum_{\alpha_\beta} \delta_{\alpha\alpha_\beta} \delta(x - x_\beta) \langle 0 | T^*[\psi^{\alpha_1}(x_1) \cdots \psi^{\alpha_{\beta-1}}(x_{\beta-1}) \psi^{\alpha_{\beta+1}}(x_{\beta+1}) \cdots \psi^{\alpha_n}(x_n)] | 0 \rangle \tag{25}$$

where the symbol T^* stands for the covariantized T product (Suura and Young, 1973). Fixing t and letting

$$t_1, t_2, \ldots, t_m \to +\infty, t_{m+1}, t_{m+2}, \ldots, t_n \to -\infty$$

and using the reduction formula (Young, 1987), we find that the expression (25) becomes

$$\left\langle \text{out}, m \middle| \left(\frac{\delta I^P}{\delta \psi^\alpha} \right) \middle| n - m, \text{in} \right\rangle = 0 \tag{26}$$

Since m and n are arbitrary, this implies

$$\frac{\delta I^P}{\delta \psi^\alpha} = 0 \tag{27a}$$

Similarly, let us only consider the transformation of the canonical momenta $\pi_\alpha(x)$; we obtain

$$\frac{\delta I^P}{\delta \pi_\alpha} = 0 \tag{27b}$$

where $\delta I^P/\delta\pi_\alpha = \dot{\psi}^\alpha - \delta H_c/\delta\pi_\alpha$. The operator expression (27) is the quantum canonical equation for a system with a regular Lagrangian.

For a system with a singular Lagrangian, due to the singularity of the Lagrangian, the motion of the system is restricted to a hypersurface of the phase space, determined by a set of constraints. Let $\Lambda_k(\psi^\alpha, \pi_a) \approx 0$ $(k = 1, 2, \ldots, K)$ be first-class constraints, and $\theta_j(\psi^\alpha, \pi_\alpha) \approx 0$ $(j = 1, 2, \ldots, J)$ be second-class constraints. The gauge conditions connecting the first-class constraints are $\Omega_k(\psi^\alpha, \pi_\alpha) \approx 0$ $(k = 1, 2, \ldots, K)$. The generating functional of the Green function for this system is given by (Gitman and Tyutin, 1990)

$$Z[J, J_1] = \int \mathcal{D}\psi^\alpha \, \mathcal{D}\pi_\alpha \prod_{j,k,l} \delta(\theta_j)\delta(\Lambda_k)\delta(\Omega_l) \det|\{\Lambda_k, \Omega_l\}|$$

$$\times [\det|\{\theta_i, \theta_j\}|]^{1/2} \exp\left\{ i \int d^4x \, (\mathcal{L}^P + J_\alpha \psi^\alpha + J_1^\alpha \pi_\alpha) \right\} \tag{28}$$

Using the integral properties of the Grassmann variables $\overline{C}(x)$ and $C(x)$, we obtain

$$\det|\{A_k(x), B_j(y)\}|$$
$$= \int \mathcal{D}C_l(x)\, \mathcal{D}\overline{C}_k(y) \exp\left[i \int d^4x\, d^4y\, \overline{C}_k(x)\{A_k(x), B_l(y)\}C_l(y)\right] \tag{29}$$

Thus, the expression (28) can be written as

$$Z[J, J_1] = \int \mathcal{D}\psi^\alpha\, \mathcal{D}\pi_\alpha\, \mathcal{D}\lambda_m\, \mathcal{D}C\, \mathcal{D}\overline{C} \exp\left\{i\left[I^P_{\text{eff}} + \int d^4x\, (J_\alpha\psi^\alpha + J^\alpha_1\pi_\alpha)\right]\right\} \tag{30a}$$

where

$$I^P_{\text{eff}} = \int d^4x\, \mathcal{L}^P_{\text{eff}} = \int d^4x\, (\mathcal{L}^P + \mathcal{L}_{Lm} + \mathcal{L}_{gh}) \tag{30b}$$

$$\mathcal{L}_{Lm} = \lambda_j\theta_j + \lambda_k\Lambda_k + \lambda_l\Omega_l \tag{30c}$$

$$\mathcal{L}_{gh} = \int d^4y\, [\overline{C}_k(x)\{\Lambda_k(x), \Omega_l(y)\}C_l(y) + \frac{1}{2}\overline{C}_i(x)\{\theta_i(x), \theta_j(y)\}C_j(y)] \tag{30d}$$

$\lambda_m = (\lambda_j, \lambda_k, \lambda_l)$ are multiplier fields. The generating functional (30a) is invariant under the translation of the canonical variables; one can still proceed in the same way to obtain the quantum canonical equations for a system with a singular Lagrangian, but in this case one must use I^P_{eff} instead of I^P in equation (27). The invariant of the generating functional under the translation of Lagrange multipliers leads to the constraint and gauge conditions.

In classical theories of constrained Hamiltonian systems, Dirac conjectured that all the first-class constraints (primary and secondary) are generators of gauge transformations. In turn, this problem is closely related to the equivalence of Dirac's procedure using the extended Hamiltonian and Lagrangian descriptions (Costa *et al.*, 1985; Cabo, 1986; Henneaux *et al.*, 1990). From time to time there have been objections to Dirac's conjecture (Li, 1993b,c). According to the above discussion we see that in quantum theories of constrained Hamiltonian systems the canonical equations of motion are derived from the canonical effective action I^P_{eff}, which involves all constraints and gauge conditions. The quantum canonical equations differ from classical ones whether Dirac's conjecture holds true or not.

4. CANONICAL WARD IDENTITY FOR CONSTRAINED HAMILTONIAN SYSTEM

As is well known, the Ward (or Ward–Takahashi) identity plays an important role in quantum field theories. The treatments for this identity usually are based on examination of the Lagrangian (or effective Lagrangian) in configuration (coordinate) space and the invariance of the generating functional of the Green function under the corresponding transformation expressed in terms of Lagrange's variables (Suura and Young, 1973; Lhallabi, 1989). In the more general case (especially for the constrained Hamiltonian system), the phase space generating functional cannot be simplified by carrying out explicit integration over momenta. Then the generating functional cannot be represented in the so-called Lagrangian form, i.e., in the form of a functional integral only over coordinates of the expression containing a certain effective Lagrangian. In certain cases, even if the integration over momenta can be carried out, the effective Lagrangians sometimes are singular (Lee and Yang, 1962; Du *et al.*, 1980). This singularity is expected to cancel in the procedure of renormalization. Therefore a discussion of symmetry in phase space for those system is necessary.

In the expression (30a), we introduce the corresponding exterior sources for λ_m, $\bar{C}$, and C fields, respectively, and denote $\phi^\alpha = (\psi^\alpha, \lambda_m, \bar{C}, C)$, $J_\alpha = (J_\alpha, U_m, \xi, \bar{\xi})$; the generating functional of the Green function for the constrained Hamiltonian system can be written as

$$Z[J_\alpha, J_1^\alpha] = \int \mathcal{D}\phi^\alpha\, \mathcal{D}\pi_\alpha \exp\{i[I_{\text{eff}}^P + J_\alpha\phi^\alpha + J_1^\alpha\pi_\alpha]\} \tag{31}$$

Let us now consider a general transformation in extended phase space; the infinitesimal transformation is given by

$$\begin{cases} x^{\mu'} = x^\mu + \Delta x^\mu = x^\mu + R_\sigma^\mu \epsilon^\sigma(x) \\ \phi^{\alpha'}(x') = \phi^\alpha(x) + \Delta\phi^\alpha(x) = \phi^\alpha(x) + S_\sigma^\alpha \epsilon^\sigma(x) \\ \pi'_\alpha(x') = \pi_\alpha(x) + \Delta\pi_\alpha(x) + T_{\alpha\sigma}\epsilon^\sigma(x) \end{cases} \tag{32}$$

where ϵ^σ ($\sigma = 1, 2, \ldots, r$) are infinitesimal arbitrary functions, whose values on the boundary of the time-space domain vanish, R_σ^μ, S_σ^α, and $T_{\alpha\sigma}$ are linear differential operators

$$R_\sigma^\mu = a_\sigma^{\mu\nu(k)}\partial_{\nu(k)}, \qquad S_\sigma^\alpha = b_\sigma^{\alpha\nu(l)}\partial_{\nu(l)}, \qquad T_{\alpha\sigma} = c_{\alpha\sigma}^{\nu(m)}\partial_{\nu(m)}$$

$$\nu(n) = \underbrace{\nu\mu\cdots\sigma\rho}_{n}, \qquad \partial_{\nu(n)} = \underbrace{\partial_\nu\partial_\mu\cdots\partial_\sigma\partial_\rho}_{n} \tag{33}$$

a, b, and c, etc., are functions of x, ϕ^α, and π_α. The variation of the canonical effective action (30b) under the transformation (32) is given by

$$\delta I^P_{\text{eff}} = \int d^4x \left\{ \frac{\delta I^P_{\text{eff}}}{\delta \phi^\alpha} \delta\phi^\alpha + \frac{\delta I^P_{\text{eff}}}{\delta \pi_\alpha} \delta\pi_\alpha + \partial_\mu[(\pi_\alpha \dot{\phi}^\alpha - \mathcal{H}_{\text{eff}})\Delta x^\mu] + \frac{d}{dt}(\pi_\alpha \delta\phi^\alpha) \right\} \tag{34a}$$

where $\mathcal{H}_{\text{eff}}$ is a canonical Hamiltonian density of the field ϕ^α, and

$$\delta\phi^\alpha = \Delta\phi^\alpha - \phi^\alpha_{,\mu}\Delta x^\mu, \qquad \delta\pi_\alpha = \Delta\pi_\alpha - \pi_{\alpha,\mu}\Delta x^\mu \tag{34b}$$

$$\frac{\delta I^P_{\text{eff}}}{\delta\phi^\alpha} = -\dot{\pi}_\alpha - \frac{\delta H_{\text{eff}}}{\delta \phi^\alpha}, \qquad \frac{\delta I^P_{\text{eff}}}{\delta \pi_\alpha} = \dot{\phi}^\alpha - \frac{\delta H_{\text{eff}}}{\delta \pi_\alpha} \tag{34c}$$

Let it be supposed that the Jacobian of the transformation (32) is $J[\phi, \pi, \epsilon]$. From the boundary conditions of $\epsilon^\sigma(x)$ and the invariant of the generating functional (31) under the transformation (32), one has

$$Z[J_\alpha, J^\alpha_1] = \int \mathcal{D}\phi^\alpha\, \mathcal{D}\pi_\alpha \left[J[\phi, \pi, \epsilon] + i\delta I^P_{\text{eff}} + i\int d^4x(J_\alpha \delta\phi^\alpha + J^\alpha_1 \delta\pi_\alpha) \right] \times \exp\left\{ i\left[I^P_{\text{eff}} + \int d^4x\,(J_\alpha\phi^\alpha + J^\alpha_1 \pi_\alpha) \right] \right\} \tag{35}$$

The invariance of the generating functional (31) implies $\delta Z/\delta\epsilon^\sigma|_{\epsilon^\alpha=0} = 0$. Differentiating (35) with respect to $\epsilon^\sigma(x)$ and setting $J_\alpha = J^\alpha_1 = 0$, one obtains

$$\int \mathcal{D}\phi^\alpha\, \mathcal{D}\pi_\alpha \left[J^0_\sigma + \tilde{S}^\alpha_\sigma \left(\frac{\delta I^P_{\text{eff}}}{\delta\phi^\alpha} \right) - \tilde{R}^\mu_\sigma \left(\phi^\alpha_{,\mu} \frac{\delta I^P_{\text{eff}}}{\delta\phi^\alpha} \right) + \tilde{T}_{\alpha\sigma} \left(\frac{\delta I^P_{\text{eff}}}{\delta\pi_\alpha} \right) - \tilde{R}^\mu_\sigma \left(\pi_{\alpha,\mu} \frac{\delta I^P_{\text{eff}}}{\delta \pi_\alpha} \right) \right] \exp(iI^P_{\text{eff}}) = 0 \tag{36}$$

where $J^0_\sigma = \delta J[\phi^\alpha, \pi, \epsilon]/\delta\epsilon^\sigma|_{\epsilon^\sigma=0}$, and $\tilde{R}^\mu_\sigma$, $\tilde{S}^\alpha_\sigma$, and $\tilde{T}_{\alpha\sigma}$ are adjoint operators with respect to R^μ_σ, S^α_σ, *and* $T_{\alpha\sigma}$, respectively (Li, 1987). The Green function connected with (36) is given by

$$\left\langle 0 \left| T^* \left[J^0_\sigma + \tilde{S}^\alpha_\sigma \left(\frac{\delta I^P_{\text{eff}}}{\delta\phi^\alpha} \right) - \tilde{R}^\mu_\sigma \left(\phi^\alpha_{,\mu} \frac{\delta I^P_{\text{eff}}}{\delta\phi^\alpha} \right) + \tilde{T}_{\alpha\sigma} \left(\frac{\delta I^P_{\text{eff}}}{\delta\pi_\alpha} \right) - \tilde{R}^\mu_\sigma \left(\pi_{\alpha,\mu} \frac{\delta I^P_{\text{eff}}}{\delta\pi_\alpha} \right) \right] \right| 0 \right\rangle \Bigg|_{\pi_\alpha = \partial\mathcal{L}/\partial\phi^\alpha} = 0 \tag{37}$$

According to the invariance of the generating functional, from (35) one can also obtain the canonical Ward identities for the constrained Hamiltonian system for the case $J = 1$:

$$\left[\tilde{S}^{\alpha}_{\sigma}\left(\frac{\delta I^{P}_{\text{eff}}}{\delta\phi^{\alpha}}\right) - \tilde{R}^{\mu}_{\sigma}\left(\phi^{\alpha}_{,\mu}\frac{\delta I^{P}_{\text{eff}}}{\delta\phi^{\alpha}}\right) + \tilde{S}^{\alpha}_{\sigma}J_{\alpha} + \tilde{T}_{\alpha\sigma}\left(\frac{\delta I^{P}_{\text{eff}}}{\delta\pi_{\alpha}}\right)\right.$$

$$\left. - \tilde{R}^{\mu}_{\sigma}\left(\pi_{\alpha,\mu}\frac{\delta I^{P}_{\text{eff}}}{\delta\pi_{\alpha}}\right) + \tilde{T}_{\alpha\sigma}J^{\alpha}_{1}\right]_{\phi^{\alpha}\to\frac{1}{i}\frac{\delta}{\delta J_{\alpha}},\pi_{\alpha}\to\frac{1}{i}\frac{\delta}{\delta J^{\alpha}_{1}}} Z[J_{\alpha}, J^{\alpha}_{1}] = 0 \tag{38}$$

Differentiating (38) with respect to the exterior sources, one can obtain the other forms of the canonical Ward identities.

In the following sections we give some preliminary applications of the above results to gauge field theories.

5. ABELIAN CASE

Consider the model with the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_{\mu}\phi - mB_{\mu})(\partial^{\mu}\phi - mB^{\mu}) \tag{39a}$$

$$F_{\mu\nu} = \partial_{\mu}B_{\nu} - \partial_{\nu}B_{\mu} \tag{39b}$$

which is functionally equivalent to the mixed Chern–Simons one (Dorey and Mavromatos, 1990). The Chern–Simons theory plays an important role in superconductivity. The momenta conjugate to $B_{\mu}(x)$ and $\phi(x)$ are

$$\pi^{\mu}(x) = \frac{\partial\mathcal{L}}{\partial\dot{B}_{\mu}(x)} = -F^{0\mu}(x) \tag{40a}$$

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)} = -mB^{0}(x) + \dot{\phi}(x) \tag{40b}$$

respectively. The primary constraint is only

$$\phi^{0} = \pi^{0}(x) \approx 0 \tag{41}$$

The canonical Hamiltonian is given by

$$H_c = \int d^3x\,\mathcal{H}_c = \int d^3x\,(\pi^{\mu}\dot{B}_{\mu} + \pi\dot{\phi} - \mathcal{L})$$

$$= \int d^3x\left[\frac{1}{2}\pi_i^2 + \frac{1}{2}\pi^2 + \frac{1}{4}F_{ik}F^{ik} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{1}{2}mB_i^2\right.$$

$$\left. - mB_i\partial_i\phi + (\partial_i\pi_i + m\pi)B^0\right] \tag{42}$$

and

$$H_T = \int d^3x\,(\mathcal{H}_c + \lambda\phi^0) \tag{43}$$

The stationarity condition of ϕ^0 yields the following secondary constraint:

$$\chi = -\partial_i\pi_i - m\pi \approx 0 \tag{44}$$

The stationarity of the secondary constraint χ does not yield any new constraint. All constraints $\Lambda_1 = \phi^0$ and $\Lambda_2 = \chi$ are of first class. Then one can construct the generator of the gauge transformation as follows:

$$\begin{aligned} G &= \int d^3x\,[\epsilon(x)(\partial_i\pi_i + m\pi) + \epsilon(x)_{,0}\pi^0] \\ &= \int d^3x\,[\pi_\mu\partial^\mu\epsilon(x) + m\pi\epsilon(x)] \end{aligned} \tag{45}$$

This G produces the following gauge transformation:

$$\begin{cases} B'_\mu(x) = B_\mu(x) + \{B_\mu, G\} = B_\mu + \partial_\mu\epsilon(x), & \pi^{\mu'}(x) = \pi^\mu(x) \\ \phi'(x) = \phi(x) + \{\phi, G\} = \phi(x) + m\epsilon(x), & \pi' = \pi(x) \end{cases} \tag{46}$$

Under the transformation (46) the canonical action is invariant.

According to the rule of path integral quantization, for each first-class constraint, one must choose a gauge condition. Consider the Coulomb gauge; $\Omega_2 = \partial_i B_i \approx 0$, the stationarity of Ω_2; one has another gauge constraint, $\Omega_1 = \partial_i\pi_i + \nabla^2 A^0 \approx 0$. It is easy to check that $\det|\{\Lambda_\alpha, \Omega_\beta\}|$ is independent of fields, and thus one can omit it from the generating functional of the Green function; then one has

$$\begin{aligned} Z[J_\mu, J_1^\mu, J, J_1, U_k, V_l] &= \int \mathcal{D}B^\mu\,\mathcal{D}\pi_\mu\,\mathcal{D}\phi\,\mathcal{D}\pi\,\mathcal{D}\mu_k\,\mathcal{D}\omega_l \\ &\times \exp\left\{i\int d^4x\,[\mathcal{L}_{\text{eff}}^{\text{P}} + J_\mu B^\mu + J_1^\mu\pi_\mu + J\phi + J_1\pi + U_k\mu_k + V_l\omega_l]\right\} \end{aligned} \tag{47a}$$

where

$$\mathcal{L}_{\text{eff}}^P = \mathcal{L}^P + \mathcal{L}_{lm} = \mathcal{L}^P + \mu_k\Lambda_k + \omega_l\Omega_l \tag{47b}$$

U_k and V_l are exterior sources corresponding to the multiplier fields $\mu_k(x)$ and $\omega_l(x)$, respectively. We denote $\phi^\alpha = (B^\mu, \phi, \mu_k, \omega_l)$, $\pi_\alpha = (\pi_\mu, \pi)$, J_α

$= (J_\mu, J, U_k, V_l)$, and $J_1^\alpha = (J^\mu, J_1)$; thus the generating functional (47a) can be written as (31).

The canonical action is invariant under the transformation (46), and the Jacobian of transformation (46) is equal to unity. The invariance of the generating functional (47a) under (46) implies the canonical Ward identity

$$\left[-\partial_0\nabla^2 \frac{\delta}{\delta V_1} + \nabla^2 \frac{\delta}{\delta V_2} + mJ - \partial_\mu J^\mu\right] Z[J, J_1] = 0 \tag{48}$$

Let $Z[J, J_1] = \exp(iW[J, J_1])$, and use the definition of $\Gamma(\phi^\alpha, \pi_\alpha]$ which is given by performing a functional Legendre transformation on $W[J, J_1]$,

$$\Gamma[\phi^\alpha, \pi_\alpha] = W[J, J_1] - \int d^4x\,(J_\alpha\phi^\alpha + J_1^\alpha\pi_\alpha) \tag{49}$$

and

$$\frac{\delta W}{\delta J_\alpha(x)} = \phi^\alpha(x), \qquad \frac{\delta\Gamma}{\delta\phi^\alpha(x)} = -J_\alpha(x) \tag{50a}$$

$$\frac{\delta W}{\delta J_1^\alpha(x)} = \pi_\alpha(x), \qquad \frac{\delta\Gamma}{\delta\pi_\alpha(x)} = -J_1^\alpha(x) \tag{50b}$$

Then (49) becomes

$$-\partial_0\nabla^2\omega_1(x) + \nabla^2\omega_2(x) - m\frac{\delta\Gamma}{\delta\phi(x)} + \partial_\mu \frac{\delta\Gamma}{\delta B_\mu(x)} = 0 \tag{51}$$

Differentiating (51) with respect to $\phi(x)$ and setting all field variables (including multiplier fields) equal to zero, one gets

$$\frac{\delta^2\Gamma[0]}{\delta\phi(x_1)\delta\phi(x_2)} = \frac{1}{m}\partial_\mu \frac{\delta^2\Gamma[0]}{\delta B_\mu(x_1)\delta\phi(x_2)} \tag{52}$$

Differentiating (51) with respect to $B_\nu(x)$ and setting all fields equal to zero, one obtains

$$\partial_\mu \frac{\delta^2\Gamma[0]}{\delta B_\mu(x_1)\delta B_\nu(x_2)} = m\frac{\delta^2\Gamma[0]}{\delta\phi(x_1)\delta B_\nu(x_2)} \tag{53}$$

Differentiating (51) with respect to $\phi(x)$ and $B_\nu(x)$ and setting all fields equal to zero, one obtains

$$\partial_\mu \frac{\delta^3\Gamma[0]}{\delta B_\mu(x_1)\delta\phi(x_2)\delta B_\nu(x_3)} = m\frac{\delta^3\Gamma[0]}{\delta\phi(x_1)\delta\phi(x_2)\delta B_\nu(x_3)} \tag{54}$$

Differentiating (51) with respect to $B_\rho(x)$ and $B_\nu(x)$ and setting all fields equal to zero, one obtains

$$\partial_\mu \frac{\delta^3\Gamma[0]}{\delta B_\mu(x_1)\delta B_\rho(x_2)\delta B_\nu(x_3)} = m \frac{\delta^3\Gamma[0]}{\delta\phi(x_1)\delta B_\rho(x_2)\delta B_\nu(x_3)} \tag{55}$$

From (54) and (55) one also gets

$$m^2 \frac{\delta^3\Gamma[0]}{\delta\phi(x_1)\delta\phi(x_2)\delta B_\nu(x_3)} = \partial_\mu\partial_\rho \frac{\delta^3\Gamma[0]}{\delta B_\mu(x_1)\delta B_\rho(x_2)\delta B_\nu(x_3)} \tag{56}$$

The expressions (52)–(56) represent some Ward identities for proper vertices. Differentiating (51) with respect to $\pi(x_2)$ and $\pi(x_3)$ and setting all fields equal to zero, one obtains

$$\partial_\mu \frac{\delta^3\Gamma[0]}{\delta B_\mu(x_1)\delta\pi(x_2)\delta\pi(x_3)} = m \frac{\delta^3\Gamma[0]}{\delta\phi(x_1)\delta\pi(x_2)\delta\pi(x_3)} \tag{57}$$

Using (40b), we can express (57) in terms of the variables in configuration space.

From the above example we see that in order to derive the canonical Ward identity for the proper vertices, one only needs to require that the canonical action is invariant under the gauge transformation in phase space. The generator of the gauge transformation can be constructed once the Hamiltonian and the first-class constraints of the theory are given. Using this gauge invariance, the canonical Ward identity in phase space is deduced immediately.

6. NON-ABELIAN CASE

The Lagrangian of the non-Abelian gauge field $B^a_\mu(x)$ coupled to a spinor field $\psi(x)$ is given by

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu} + \bar{\psi}[i\gamma^\mu(\partial_\mu - igB^a_\mu T^a) - m]\psi \tag{58a}$$

$$F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu - gf^a_{bc}B^b_\mu B^c_\nu \tag{58b}$$

where T^a are generators of gauge group $\underline{G}$, and f^a_{bc} are structure constants of group G. The canonical momenta of ψ, $\bar{\psi}$, and B^a_μ are

$$\pi = i\bar{\psi}\gamma_0, \qquad \bar{\pi} = 0, \qquad \pi^\mu_a = -F^{0\mu}_a \tag{59}$$

respectively. The canonical Hamiltonian of this system is given by

$$\begin{aligned} H_c = \int d^3x\, \mathcal{H}_c = \int d^3x \bigg[& \frac{1}{2}\pi^a_i\pi^a_i - B^a_0(D^a_{ib}\pi^i_b - ig\pi T^a\psi) + \frac{1}{4}F^a_{ij}F^{ij}_a \\ & + \frac{1}{2}\pi\gamma^0\gamma^i(\partial^i\psi) - \frac{1}{2}(\partial^i\pi^k)\gamma^0\gamma^i\psi \\ & - ig\pi\gamma^0\gamma^i\psi B^a_i T_a - im\pi\gamma^0\psi \bigg] \end{aligned} \tag{60}$$

where

$$D^a_{b\mu} = \delta^a_b\partial_\mu + gf^a_{bc}B^c_\mu \tag{61}$$

The primary constraints are

$$\theta_1 = \pi - i\bar{\psi}\gamma_0 \approx 0 \tag{62}$$

$$\theta_2 = \bar{\pi} \approx 0 \tag{63}$$

$$\Lambda^a_1 = \pi^0_a \approx 0 \tag{64}$$

The total Hamiltonian is given by

$$H_T = \int d^3x\,(\mathcal{H}_c + \lambda_1\theta_1 + \lambda_2\theta_2 + \mu^a_1\Lambda_{1a}) \tag{65}$$

The stationarity condition of θ_j ($j = 1, 2$), $\{\theta_j, H_T\} \approx 0$, yields the equation to determine the Lagrange multipliers λ_j ($j = 1, 2$). The stationarity of Λ^a_1 yields the secondary constraints

$$\chi^a = D^a_{ib}\pi^i_b - ig\pi T_a\psi \approx 0 \tag{66}$$

Let

$$\Lambda^a_2 = \chi^a + igT^a(\theta_1\psi - \bar{\psi}\theta_2) \approx 0 \tag{67}$$

One can easily check that the constraints Λ^a_1 and Λ^a_2 are of first class, and the constraints θ_1 and θ_2 are of second class.

The gauge conditions can be chosen as (Sundermyer, 1982)

$$\Omega^a_1 = \partial^i\pi^a_i + M^{ab}B^a_b \approx 0 \tag{68}$$

$$\Omega^a_2 = \partial^i A^a_i \approx 0 \tag{69}$$

The factors $\det|\{\theta_i(x), \theta_j(x)\}|$ are independent of field variables, which may be omitted from the integrand, and (Sundermyer, 1982)

$$\det|\{\Lambda^a_i, \Omega^b_j\}| = \det|M^{ab}\delta(\mathbf{x} - \mathbf{y})| = \det M_c \tag{70}$$

Thus, the generating functional can be written as

$$\begin{aligned} &Z[J, J_1, \bar{\xi}, \xi_1, \xi, \bar{\xi}_1, \bar{\zeta}, \zeta, U, X, Y] \\ &= \int \mathcal{D}B^a_\mu\, \mathcal{D}\pi^\mu_a \mathcal{D}\psi \mathcal{D}\pi\, \mathcal{D}\bar{\psi}\mathcal{D}\bar{\pi}\mathcal{D}\bar{C}\mathcal{D}C\mathcal{D}\mu\mathcal{D}\omega\mathcal{D}\nu \\ &\quad\times \exp\left\{i\int d^4x\,[\mathcal{L}^P_{\text{eff}} + J^\mu_a A^a_\mu + J^a_{1\mu}\pi^\mu_a + \bar{\xi}\psi + \pi\xi_1 \right. \\ &\quad\left. + \bar{\psi}\xi + \bar{\xi}_1\bar{\pi} + \bar{\zeta}_a C^a + \bar{C}^a\zeta_a + U_k\mu_k + X_l\omega_l + Y_i\nu_i]\right\} \end{aligned} \tag{71a}$$

where

$$\mathcal{L}_{\text{eff}}^{P} = \mathcal{L}^{P} + \mathcal{L}_{Lm} + \mathcal{L}_{gh} \tag{71b}$$

$$\mathcal{L}^{P} = \pi\dot{\psi} + \pi_a^{\mu}\dot{A}_{\mu}^{a} - \mathcal{H}_c \tag{71c}$$

$$\mathcal{L}_{Lm} = \nu_j\theta_j + \mu_k^a\Lambda_k^a + \omega_l^a\Omega_l^a \tag{71d}$$

$$\mathcal{L}_{gh} = \overline{C}_a M^{ab} C_b = -\partial^i\overline{C}^a D_{ib}^a C^b \tag{71e}$$

Let us denote $\phi^{\alpha} = (B_{\mu}^{a}, \psi, \overline{\psi}, C, \overline{C}, \mu, \omega, \nu)$, $\pi_{\alpha} = (\pi_a^{\mu}, \pi, \overline{\pi})$, $J_{\alpha} = (J_{\mu}^{a}, \overline{\xi}, \xi, \overline{\zeta}, \zeta, U_k, X_l, Y_j)$, $J_1^{\alpha} = (J_{1\mu}^{a}, \xi_1, \overline{\xi}_1)$; the expression (71a) can be written as (31).

Let us list the vector, axial-vector, and pseudoscalar currents:

$$\begin{aligned} V_{\mu}^{a}(x) &= \overline{\psi}(x)\gamma_{\mu}T^{a}\psi(x) \\ A_{\mu}^{a}(x) &= \overline{\psi}(x)\gamma_{\mu}\gamma_5 T^{a}\psi(x) \\ P^{a}(x) &= \overline{\psi}(x)i\gamma_5 T^{a}\psi(x) \end{aligned} \tag{72}$$

respectively. For example, if the gauge group G is $SU(3)$, then $T^{a} = 1/2\lambda^{a}$, where λ^{a} $(a = 1, 2, \ldots, 8)$ are Gell-Mann matrices. The generating functional with extended exterior sources is given by

$$\begin{aligned} Z[J, J_1, \upsilon, a, p] = \int \mathcal{D}\phi^{\alpha}\, \mathcal{D}\pi_{\alpha} \exp\Big\{ i \int d^4x\, [\mathcal{L}_{\text{eff}}^{P} + J_{\alpha}\phi^{\alpha} \\ + J_1^{\alpha}\pi_{\alpha} + \upsilon_a^{\mu}V_a^{\mu} + a_a^{\mu}A_{\mu}^{a} + p_a P^{a}] \Big\} \end{aligned} \tag{73}$$

where υ_a^{μ}, a_a^{μ}, and p_a are exterior sources with respect to V_{μ}^{a}, A_{μ}^{a}, and P^{a}, respectively. Consider a transformation of

$$\begin{aligned} \psi'(x) &= [1 + \epsilon^{a}(x)\gamma_5 T^{a}]\psi(x), &\quad \pi'(x) &= \pi(x)[1 - i\epsilon^{a}(x)\gamma_5 T^{a}] \\ \overline{\psi}'(x) &= \overline{\psi}(x)[1 + i\epsilon^{a}(x)\gamma_5 T^{a}], &\quad \overline{\pi}'(x) &= [1 - i\epsilon^{a}(x)\gamma_5 T^{a}]\overline{\pi}(x) \end{aligned} \tag{74}$$

The variation of canonical I^{P} under the transformation (74) is given by

$$\delta I^{P} = \int d^4x\, \epsilon^{a}(x)[\partial^{\mu}A_{\mu}^{a} - 2mP^{a} - gf_{bc}^{a}A_{\mu}^{b}B_{c}^{\mu}] \tag{75}$$

The change of $\Lambda_2^{b}(x)$ under the transformation (74) is given by $\delta\Lambda_2^{b} = \Lambda_{2a}^{b}(\psi, \overline{\psi}, \overline{\pi})\epsilon^{a}(x)$. The Jacobian of the transformation (74) is equal to unity

(Suura and Young, 1973). Thus, the invariance of the generating functional (73) under the transformation (74) implies that

$$\int \mathcal{D}\phi^\alpha\, \mathcal{D}\pi_\alpha\, [\partial^\mu A^a_\mu - 2mP^a(x) - gf^a_{bc}A^b_\mu B^\mu_c + U_2\mu^b_2\Lambda^b_{2a} + i\bar{\xi}\gamma_5 T^a\psi$$
$$- i\gamma_5 T^a\pi\bar{\xi}_1 + iT^a\bar{\psi}\gamma_5\xi - iT^a\bar{\pi}\gamma_5\xi_1 + v^\mu_a\delta V^a_\mu + a^\mu_a\delta A^a_\mu + p_a\delta P^a]$$
$$\times \exp\left\{i\int d^4x\, [\mathcal{L}^P_{\text{eff}} + J_\alpha\phi^\alpha + J^\alpha_1\pi_\alpha + v^\mu_a V^a_\mu + a^\mu_a A^a_\mu + p_a P^a]\right\} = 0 \tag{76}$$

Setting all exterior sources (including U_2) equal to zero, from (76) one obtains

$$\langle 0|T^*[\partial^\mu A^a_\mu(x) - 2mP^a(x) - gf^a_{bc}A^b_\mu(x)B^\mu_c(x)]|0\rangle = 0 \tag{77}$$

This is an expression for the partial conservation of axial-vector current (PCAC). Differentiating (76) with respect to $v^b_\nu(y)$ and $v^c_\lambda(z)$ and setting all exterior sources equal to zero, one also obtains

$$\begin{aligned}&\langle 0|T^*[(\partial^\mu A^a_\mu(x) - 2mP^a(x) - gf^a_{de}A^d_\mu(x)B^\mu_e(x))V^b_\nu(y)V^c_\lambda(z)]|0\rangle\\ &= i\delta^4(x-y)f^a_{be}\langle 0|T[A^e_\nu(x)V^c_\lambda((y))]|0\rangle\\ &+ i\delta^4(x-z)f^a_{ce}\langle 0|T[A^e_\nu(x)V^b_\lambda(y)]|0\rangle\end{aligned} \tag{78}$$

This is just the naive axial-vector Ward–Takahashi identity for the AVV. Suura and Young (1973) discussed the case $f^a_{bc} = 0$, but the constraints were ignored. Here a complete and correct discussion is made from another point of view.

As is well known, the usual BRS transformation is nonlinear for the ghost fields. Here we find that the Lagrangian $\mathcal{L}^P(x) + \mathcal{L}_{gh}(x)$ is invariant under the following linear transformation:

$$\begin{cases}\psi'(x) = \psi(x) + ig\epsilon^\sigma(x)T^\sigma\psi(x), & \pi'(x) = \pi(x) - ig\pi(x)\epsilon^\sigma(x)T^\sigma\\ \bar{\psi}'(x) = \bar{\psi}(x) - ig\bar{\psi}(x)\epsilon^\sigma(x)T^\sigma, & \bar{\pi}'(x) = \bar{\pi}(x) + ig\epsilon^\sigma(x)T^\sigma\bar{\pi}(x)\\ A^{a\prime}_\mu(x) = A^a_\mu(x) + D^a_{\sigma\mu}\epsilon^\sigma(x), & \pi^{\mu\prime}_a(x) = \pi^\mu_a(x) + gf^a_{\sigma c}\pi^\mu_c(x)\epsilon^\sigma(x)\\ C^{a\prime}(x) = C^a(x) + ig(L^\sigma)_{ab}\epsilon^\sigma(x)C^b(x) & \\ \bar{C}^{a\prime}(x) = \bar{C}^a(x) - ig\bar{C}^b(x)(L^\sigma)_{ba}\epsilon^\sigma(x) + \dfrac{ig}{\Box}\partial_\mu[\bar{C}^b(x)(L^\sigma)_{ba}\partial^\mu\epsilon(x)] & \end{cases} \tag{79}$$

where L^σ ($\sigma = 1, 2, \ldots, n$) are representation matrices of the generator of the gauge group in n-dimensional space. The change of $\mathscr{L}_{Lm}$ up to a divergence term under the transformation (79) is given by

$$\delta\mathscr{L}_{Lm} = F_\sigma \epsilon^\sigma(x) \tag{80a}$$

$$\begin{aligned} F_\sigma = {} & \nu_2\Theta_{2\sigma} + \mu_1^a\Lambda_{1\sigma}^a + \mu_2^a\Lambda_{2\sigma}^a - \partial_i(\mu_2^a\Lambda_{2\sigma}^{ai}) - \partial_0\nabla^2(\omega_1^a\Omega_\sigma^{a0}) \\ & + \nabla^2(\omega_1\Omega_{1\sigma}^a) + \partial_0\partial_i(\omega_1^a\Omega_{1\sigma}^{ai0}) - \partial_i(\omega_1^a\Omega_{1\sigma}^{ai}) + \omega_1^a\Omega_{1\sigma}^{a1} \\ & + \nabla^2(\omega_2^a\Omega_{2\sigma}^a) - \partial_i(\omega_2^a\Omega_2^a) + \omega_2^a\Omega_{2\sigma}^{a0} \end{aligned} \tag{80b}$$

where $\Theta_{2\sigma}$, $\Lambda_{i\sigma}^a$, $\Omega_{i\sigma}^a$, etc., are functions of the canonical variables of the fields and their derivatives. The invariance of the generating functional (71a) under the transformation (79) implies that

$$\begin{aligned} \Bigg[& J_\sigma^0 + iF_\sigma + ig\bar{\xi}T^\sigma \frac{\delta}{\delta\bar{\xi}} - ig\xi_1 T^\sigma \frac{\delta}{\delta\xi_1} - ig\xi T^\sigma \frac{\delta}{\delta\xi} + ig\bar{\xi}_1 T^\sigma \frac{\delta}{\delta\bar{\xi}_1} \\ & - i\partial_\mu J_\sigma^\mu + f_{\sigma c}^a J_a^\mu \frac{\delta}{\delta J_c^\mu} + gf_{\sigma c}^a J_{1\mu}^a \frac{\delta}{\delta J_{1\mu}^c} + ig\bar{\xi}_a(L^\sigma)_{ab} \frac{\delta}{\delta\bar{\xi}_\sigma} \\ & - ig\zeta_a(L^\sigma)_{ba} \frac{\delta}{\delta\zeta_b} + ig\partial^\mu\left[\partial_\mu\left(\zeta_a \frac{1}{\Box}\right)(L^\sigma)_{ba} \frac{\delta}{\delta\zeta_b}\right]\Bigg] Z[J, J_1] = 0 \end{aligned} \tag{81}$$

where J_σ^0 are independent of the field variables. Introducing $W[J, J_1]$ and $\Gamma[\phi^\alpha, \pi_\alpha]$, one can still proceed in the same way as in the Abelian case. The expression can be reduced to

$$\begin{aligned} & J_\sigma^0 + iF_\sigma - ig\psi T^\sigma \frac{\delta\Gamma}{\delta\psi} + ig\pi T \frac{\delta\Gamma}{\delta\pi} + ig\bar{\psi}T \frac{\delta\Gamma}{\delta\bar{\psi}} - ig\bar{\pi}T \frac{\delta\Gamma}{\delta\bar{\pi}} \\ & + i\partial_\mu \frac{\delta\Gamma}{\delta B_\mu^\sigma} - gf_{\sigma c}^a B_\mu^c \frac{\delta\Gamma}{\delta B_\mu^a} - gf_{\sigma c}^a \pi_c^\mu \frac{\delta\Gamma}{\delta\pi_a^\mu} - igC^a(L^\sigma)_{ab} \frac{\delta\Gamma}{\delta C^b} \\ & + ig\bar{C}^a(L^\sigma)_{ba} \frac{\delta\Gamma}{\delta\bar{C}^b} - ig\partial^\mu\left[\partial_\mu\left(\frac{\delta\Gamma}{\delta\bar{C}^a}\frac{1}{\Box}\right)(L^\sigma)_{ba}\bar{C}^b\right] = 0 \end{aligned} \tag{82}$$

Differentiating (82) with respect to $B_\mu^a(x_2)$ and $B_\lambda^b(x_3)$ and setting all fields (including the multiplier fields) equal to zero, one obtains

$$\partial_\mu \frac{\delta^3\Gamma[0]}{\delta B_\mu^\sigma(x)\delta B_\nu^a(x_2)\delta B_\lambda^b(x_3)} = igf_{\sigma a}^\rho \delta(x_1 - x_2) \frac{\delta^2\Gamma[0]}{\delta B_\nu^\rho(x_1)\delta B_\lambda^b(x_3)} \tag{83}$$

This formulation to obtain the Ward identity for proper vertices has a significant advantage in that one does not need to carry out the integration over momenta in the generating functional.

Differentiating (82) with respect to $C^e(x_2)$ and $\overline{C}^f(x_3)$ and setting all fields equal to zero, one obtains

$$(L^\sigma)_{eb}\delta(x_1 - x_2)\frac{\delta^2\Gamma[0]}{\delta\overline{C}^f(x_3)\delta C^b(x_1)} - (L^\sigma)_{fb}\delta(x_1 - x_3)\frac{\delta^2\Gamma[0]}{\delta\overline{C}^b(x_1)\delta C^e(x_2)}$$
$$+ \partial^\mu_{x_1}\frac{\delta^3\Gamma[0]}{\delta\overline{C}^f(x_3)\delta C^e(x_2)\delta B^\mu_\sigma(x_1)} \tag{84}$$
$$+ \partial^\mu\left[\partial_\mu\left(\frac{\delta^2\Gamma[0]}{\delta\overline{C}^a(x_1)\delta C^e(x_2)}\frac{1}{\Box}\right)(L^\sigma)_{af}\delta(x_1 - x_3)\right] = 0$$

This new form of the Ward identity for the gauge-ghost field proper vertices differs from the usual Ward–Takahashi identity arising from the BRS invariance.

7. CONCLUSION

The canonical symmetry properties for a system with a singular Lagrangian have been studied. We developed an algorithm for the construction of the gauge generator of such systems once the Hamiltonian and the constraints are given. The canonical Ward identity for the constrained Hamiltonian system has been deduced. Applying our formulation to the Abelian and non-Abelian gauge theories, we obtained the PCAC and some Ward identities for proper vertices. Using the canonical Ward identity to derive those relations has a significant advantage in that one does not need to carry out the integration over momenta as in the traditional treatment in configuration space. Moreover, one only requires the $\mathcal{L}^P$ and $\mathcal{L}_{gh}$ to be invariant under the (nonlocal) transformation in non-Abelian gauge theory, which differs from usual BRS invariance.

APPENDIX

We present a transformation under which the $\partial^\mu\overline{C}^a D^a_{b\mu}C^b$ is invariant. The change of $D^a_{b\mu}C^b$ is

$$D^{a\prime}_{b\mu}C^b = D^a_{b\mu}C^b + ig(L^\sigma)_{ab}\epsilon^\sigma(x)D^b_{e\mu}C^e \tag{A1}$$

under the following transformation:

$$C^{a\prime} = C^a + ig(L^\sigma)_{ab}\epsilon^\sigma(x)C^b$$
$$A^{a\prime}_\mu = A^a_\mu + D^a_{\sigma\mu}\epsilon^\sigma(x)$$

Thus if we further consider a transformation of $\overline{C}^a$:

$$\partial^\mu\overline{C}^{a\prime} = \partial^\mu\overline{C}^a - ig\partial^\mu\overline{C}^b(L^\sigma)_{ba}\epsilon^\sigma(x)$$

then the $\partial^{\mu}\overline{C}^{a}D^{a}_{b\mu}C^{b}$ is invariant under the following transformation:

$$\begin{cases} C^{a\prime}(x) = C^{a}(x) + ig(L^{\sigma})_{ab}\epsilon^{\sigma}(x)C^{b}(x) \\ \overline{C}^{a\prime}(x) = \overline{C}^{a}(x) - ig\overline{C}^{b}(x)(L^{\sigma})_{ba}\epsilon^{\sigma}(x) + \frac{ig}{\Box}\,\partial_{\mu}[\overline{C}^{b}(L^{\sigma})_{ba}\partial^{\mu}\epsilon^{\sigma}(x)] \\ A^{a\prime}_{\mu}(x) = A^{a}_{\mu}(x) + D^{a}_{\sigma\mu}\epsilon^{\sigma}(x) \end{cases} \quad \text{(A2)}$$

The second equation of (A2) can be also written as

$$\overline{C}^{a\prime}(x) = \overline{C}^{a}(x) - ig\overline{C}^{b}(x)(L^{\sigma})_{ba}\epsilon^{\sigma}(x) + g\int d^4y\ \Delta_0(x, y)\partial_{\mu}[\overline{C}^{b}(y)(L^{\sigma})_{ba}\partial^{\mu}\epsilon^{\sigma}(y)] \quad \text{(A3)}$$

where

$$\Box\Delta_0(x, y) = i\delta^4(x - y) \quad \text{(A4)}$$

Thus, (A2) is a nonlocal transformation.

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